Signal Analysis & Communication ECE 355

Ch. 3.6 Discrete Time Fourier Series (DTFS)

Lecture 23_01

01-11-2023

Ch. 3.6 DISCRETE TIME FOURIER SERIES

- Fourier Series representation of DT Periodic signals
- Almost the same as that of CT Periodic signals with a few important differences.
  1. Finite Series.
  2. Therefore, no mathematical issue of convergence.
- Recall CTFS

$$x_{(t)} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad \text{— (A)}$$

where $$a_k = \frac{1}{T}\int_T x_{(t)} e^{-jk\omega_0 t}\, dt \quad \text{— (B)}$$

For CT Periodic signals.

- For DT Periodic signals, recall that DT sig is periodic with fundamental period 'N' if

$$x_{[n]} = x_{[n+N]}$$

where

fundamental freq. $\omega_0 = \frac{2\pi}{N}$

- Consider harmonically related DT complex exponential signals:

$$e^{jk\omega_0 n} = e^{jk\frac{2\pi}{N}n}, \quad k = 0, \pm 1, \pm 2, \dots \quad \text{— (1)}$$

NOTE: For DT Periodic.

1. $e^{jk\frac{2\pi}{N}n}$ is periodic in 'n' with period 'N'
2. $e^{jk\frac{2\pi}{N}n}$ is periodic in 'k' with period 'N'

k: in freq.-domain

$$e^{jk\frac{2\pi}{N}n} = e^{j(k+N)\frac{2\pi}{N}n} = e^{j(k+2N)\frac{2\pi}{N}n} = \dots$$

$$= e^{jk\frac{2\pi}{N}n + jk2\pi n} = e^{jk\frac{2\pi}{N}n}\underbrace{e^{jk2\pi n}}_{=1} = e^{jk\frac{2\pi}{N}n}.$$

"Repeated Harmonic Components"

- This is consequence of the fact that DT Complex Exponentials which differ in frequencies by a multiple of 2π are identical.

$$* \; e^{j\omega_0 n} = e^{j(\omega_0 + 2\pi)n}$$

- We now consider the representation of more general periodic signals in terms of linear combination of the signals in eqn ①

$$x_{[n]} = \underbrace{\sum_{k=\langle N\rangle}}\, a_k e^{jk\omega_0 n} \quad \text{— ②}$$

IMPORTANT ② ⇒ Here, the summation is only over a period 'N'.

Since $e^{jk\omega_0 n}$ are distinct only over a range of 'N' successive values of 'k', the summation only includes terms over this range, for e.g., $0, 1, 2, \dots, N-1$ if beginning with $k=0$.
No CONVERGENCE ISSUE — Finite

- Eqn ② is called Discrete Time Fourier Series (Synthesis) & the coefficients a_k are called DTFS coefficients.

- The Analysis eqn. of DTFS is given as:

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} \underbrace{x_{[n]}} e^{-jk\omega_0 n} \quad \text{— ③}$$

Following eqn Ⓑ (Analysis of CTFS for CT Periodic Signals)
* By replacing $\int$ with Σ as the sig is DT periodic.

Proof:

$$RHS = \frac{1}{N}\left[\sum_{n=\langle N\rangle} \sum_{m=\langle N\rangle} \underbrace{a_m e^{jm\omega_0 n} \cdot e^{-jk\omega_0 n}}\right]$$

$$= \frac{1}{N}\sum_{m=\langle N\rangle} a_m \underbrace{\sum_{n=\langle N\rangle} e^{j(m-k)\omega_0 n}}_{*}$$

* Fact:

$$\sum_{n=\langle N\rangle} e^{jk\frac{2\pi}{N}n} = \begin{cases} N, & k = 0, \pm N, \pm 2N \dots \\ 0, & \text{otherwise.} \end{cases}$$

$\boxed{\text{Ref.: Problem 3.54 Textbook.}}$

✓ Generally $(0 \to N-1)$ $\langle N \rangle$

- Using this fact:

$$* \quad RHS = \frac{1}{N} \sum_{m=\langle N \rangle} a_m \underbrace{\sum_{n=\langle N \rangle} e^{j(m-k)\omega_0 n}}$$

$$= \begin{cases} N, & \text{if } k=m \\ 0, & \text{if } k \neq m \end{cases}$$

$$= \frac{1}{N} \sum_{m=\langle N \rangle} a_m N \delta_{[m-k]}$$

$$= \frac{1}{N} \sum_{m=\langle N \rangle} a_m N \delta_{[m-k]}$$

$$= \frac{1}{N} a_k N = a_k \quad LHS$$

- Finally we have DTFS pair as:

$$x_{[n]} = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n} \quad \text{— I SYNTHESIS}$$

$$a_k = \frac{1}{N} \sum_{k=\langle N \rangle} x_{[n]} e^{-jk\omega_0 n} \quad \text{— II ANALYSIS.}$$

NOTE:

1. a_k – Spectral coefficients of $x_{[n]}$.
   They specify a decomposition of $x_{[n]}$ into sum of 'N' harmonically related complex exponentials.
2. a_k is periodic with period N. $a_k = a_{k+N}$ | can easily be proved using II
3. a_k is defined for $-\infty < k < \infty$, we only need a_k for one period to represent DT Periodic sig.

Example

$$x[n] = \sin(\theta n)$$

$$= \frac{1}{j2}\left[e^{j\theta n} - e^{-j\theta n}\right]$$

$$= -\frac{j}{2} e^{j\theta n} + \frac{j}{2} e^{-j\theta n} \quad \text{— III}$$

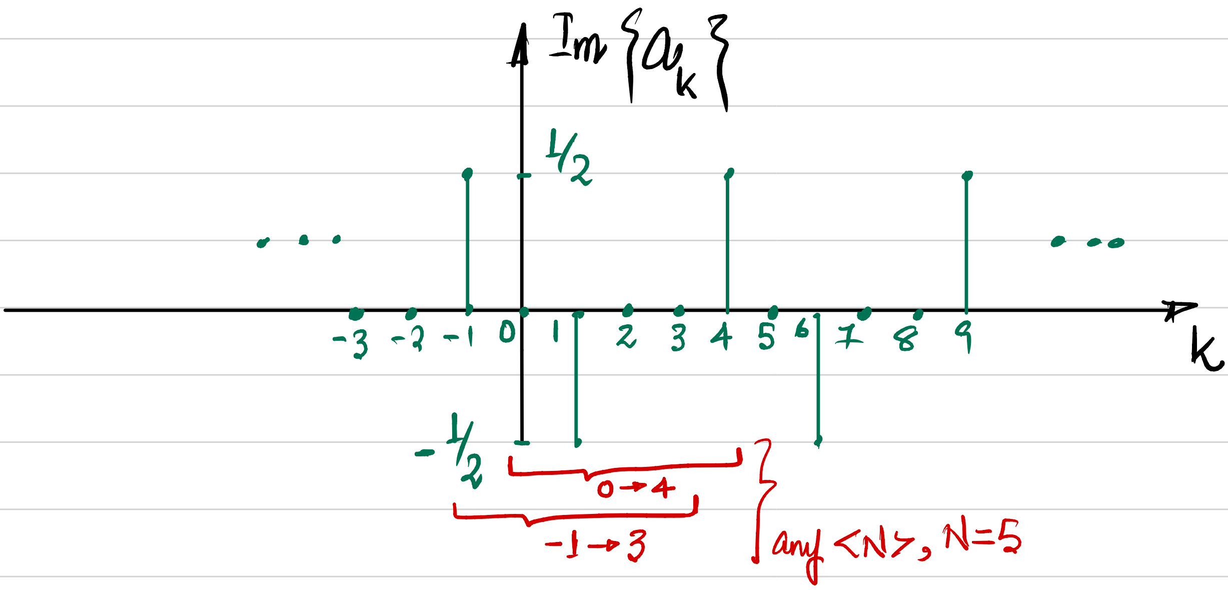
I. Suppose $\theta = \frac{2\pi}{5}$

$$x[n] = -\frac{j}{2} e^{j\frac{2\pi}{5}n} + \frac{j}{2} e^{-j\frac{2\pi}{5}n} \quad \text{— IV}$$

*Recall $e^{j\omega n}$ is periodic with period N if the frequency 'ω' is an integral multiple of $2\pi/N$ ($k \cdot 2\pi/N$; $k, N \in \mathbb{Z}$)

Comparing eqn IV with I and noting $N = 5$, $k = \pm 1$

$$\Rightarrow \underbrace{a_1 = \frac{-j}{2}, \quad a_{-1} = \frac{j}{2}}_{\text{in the same period}} \quad (\underbrace{a_0 = a_2 = a_3 = 0}_{\text{Note } N=5})$$



II. Suppose $\theta = 2\pi\left(\frac{3}{5}\right)$

eqn. (III) $\Rightarrow$ $$x_{[n]} = \frac{-j}{2} e^{j3\left(\frac{2\pi}{5}\right)n} + \frac{j}{2} e^{-j3\left(\frac{2\pi}{5}\right)n} \quad \text{— (V)}$$

Comparing eqn (V) with eqn (I) & noting $N=5$, $k=\pm 3$

$$a_3 = \frac{-j}{2}, \quad a_{-3} = \frac{j}{2}$$

Note a_3 & a_{-3} are not in the same period as $N=5$

- Let's consider the period starting from $k=0$:

$\langle N \rangle$
$0 \to 4$

$$\begin{cases} a_0 = 0 \\ a_1 = 0 \\ a_2 = a_{2-5} = a_{-3} = j/2 \\ a_3 = -j/2 \\ a_4 = 0 \end{cases}$$

a_k is periodic with period N ($N=5$ here)

- & then it repeats.