Signal Analysis & Communication ECE 355

Ch. 2.4: Linear Constant Coefficient Differential/ Difference Equation

Lecture 12

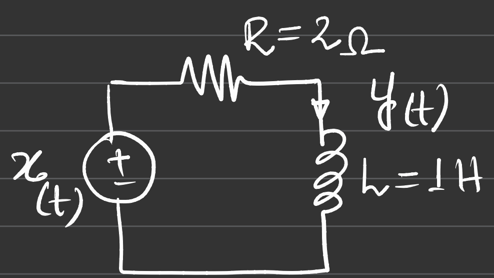
04-10-2023

Ch. 2.4. LINEAR CONSTANT COEFFICIENT DIFFERENTIAL / DIFFERENCE EQUATION (LCCDE)

- An important class of CT/DT systems is that for which the input & output are related through LCCDE.
- LCCDE describes wide variety of physical phenomenon.
  - Electrical Circuit RLC.
  - mechanical system - mass & spring with applied force.] CT
  - Accumulation of saving in bank account] DT
- We can $\boxed{\text{solve LCCDE}}$ to find EXPLICIT expressions between input & output.

NOTE: later we will see another approach, very easy & the preferred way to compute outputs for these systems.

- Example
  - Consider an RL Circuit:



  - Can be described by LCCDE

$$L\frac{dy(t)}{dt} + Ry(t) = x(t)$$

$$\frac{dy(t)}{dt} + 2y(t) = x(t) \quad - ①$$

  - Let's solve this LCCDE for finding $y(t)$ with given conditions:

$$x(t) = e^{3t}u(t) \quad \& \quad y(t) = 0 \text{ for } t<0$$

$$y(t) = ? \quad \text{for } t>0$$

- From the calculus course, we know that

$$y_{(t)} = y_{h(t)} + y_{p(t)} \quad — ②$$

- $y_{h(t)}$: homogeneous solution, a solution of:

$$\frac{dy_{(t)}}{dt} + 2y_{(t)} = 0 \quad — ③$$

"Natural Response"

- $y_{p(t)}$: particular solution, a special solution due to the I/P.
"Forced Response".

- To find $y_{h(t)}$, guess the solution of the form:

$$y_{h(t)} = Ae^{st}, \quad \text{for all 't'}$$

* Guessed exp. as its diff./int. is of exp. form as well.

- Inserting in eqn ③

$$Ase^{st} + 2Ae^{st} = 0$$

$$Ae^{st}(s+2) = 0$$

$$s = -2$$

$$\therefore y_{h(t)} = Ae^{-2t} \quad (\text{for any } A) \quad — ④$$

- Now to find $y_{p(t)}$, guess

$$y_{p(t)} = Be^{3t} \quad \text{for } t > 0 \quad — ⑤$$

* a sig. of the same form as the given I/P.

- Inserting in eqn① for finding $y_{p(t)}$

$$3Be^{3t} + 2Be^{3t} = e^{3t}$$

$$3B + 2B = 1$$

$$B = 1/5$$

- $\therefore$ eqn. (5) $\Rightarrow$ $y_{p(t)} = \frac{1}{5}e^{3t}$ for $t>0$ — (6)

- Substituting eqn. (4) & (6) in eqn (2)

$$y_{(t)} = Ae^{-2t} + \frac{1}{5}e^{3t} \quad \text{— (7)}$$

- To determine (A), we use the initial condition $y_{(0)} = 0$ for $t<0$

Eqn. (7) $\Rightarrow$ $0 = A + \frac{1}{5}$

$$A = -\frac{1}{5}$$

- Therefore, eqn. (7) $\Rightarrow$ $y_{(t)} = -\frac{1}{5}e^{-2t} + \frac{1}{5}e^{3t} \quad t>0$

NOTE

Suppose the initial condition was $y_{(0)} = 1$, then,

eqn (7) $\Rightarrow$ $1 = A + \frac{1}{5}$

$$A = \frac{4}{5}$$

$\therefore$ eqn (7) $\Rightarrow$ $y_{(t)} = \frac{4}{5}e^{-2t} + \frac{1}{5}e^{3t}, \quad t>0$ — (8)

But in this case the initial rest condition is not satisfied for the considered LTI systems, i.e.,

$$y_{(t)} = 0 \text{ for } t<t_0 \quad \text{if} \quad x_{(t)} = 0 \text{ for } t<t_0 \quad \Big| \quad \forall\, t, t_0$$

provides close approximation to most LTI systems!

For the (Causal) LTI systems, we consider in this course, the initial rest condition should be satisfied!

- The system defined by eqn ⑧ is either not causal or not an LTI.

* Remember (ref. - Lecture 10):

- Causality also ensures time invariancy of LTI systems.
- That is, with zero initial values, the response at different times will be the same (with the same I/p) [assuming R & C do not change over time].

General Form of LCCDE (CT)

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y_{(t)} = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x_{(t)} \quad \text{— Ⓐ}$$

Standard Solution:

$$y_{(t)} = y_{h(t)} + y_{p(t)}$$

Use roots of charac. Eqn.

Guess!

$x_{(t)}$	$y_{p(t)}$
e^{st}	Be^{st}
$\cos(\omega t)$	$A\cos(\omega t) + B\sin(\omega t)$
⋮	⋮

Auxilliary conditions: (Initial Conditions - Rest)

$$y_{(t_0)} = \frac{d}{dt} y_{(t_0)} = \cdots = \frac{d^{N-1}}{dt^{N-1}} y_{(t_0)} = 0$$

(assume continuity of $y(t)$ & its derivatives)

NOTE: We will see later that solving LCCDE is more straight forward in freq. domain.

LCCDE (DT)

- Recall that the differential in CT is replaced with the difference in DT.
- This implies that,

$$\frac{d}{dt}y(t) \longrightarrow d^1 y[n] \triangleq y[n] - y[n-1]$$

$$\frac{d^2 y(t)}{dt^2} \longrightarrow d^2 y[n] \triangleq dy[n] - dy[n-1]$$

$$= (y[n] - y[n-1]) - (y[n-1] - y[n-2])$$

$$= y[n] - 2y[n-1] + y[n-2] \quad \text{— (B)}$$

$$\vdots$$

$$d^k y[n] = d^{k-1} y[n] - d^{k-1} y[n-1]$$

- Therefore, the DT analogous of eqn (A) can be written as:

$$\hat{a}_0 y[n] + \sum_{k=1}^{N} \hat{a}_k d^k y[n] = \hat{b}_0 x[n] + \sum_{k=1}^{M} \hat{b}_k d^k x[n] \quad \text{— (C)}$$

- Equivalenty (using eqn (B), eqn (C) can be expressed as:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \quad \text{— (D)}$$

Standard Solution

I. Similar to the CT case

$$y[n] = y_h[n] + y_p[n]$$

- assume initial rest conditions for Causal LTI systems, that is,

$$\left.\begin{array}{ll} y[n] = 0 & \text{for } n < n_0 \\ \text{if } x[n] = 0 & \text{for } n < n_0 \end{array}\right| \forall n, n_0$$

II. We have eqn ⓓ as

$$a_0 y[n] + \sum_{k=1}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$y[n] = \frac{1}{a_0}\left[\sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k]\right]$$

- And then compute recursively!

Example

2.15 (textbook)

$$y[n] - \frac{1}{2} y[n-1] = x[n], \quad x[n] = K\delta[n]$$

$$y[n] = x[n] + \frac{1}{2} y[n-1]$$

$$y[0] = x[0] + \frac{1}{2} y[-1] = K + 0 = K$$

$$y[1] = x[1] + \frac{1}{2} y[0] = 0 + \frac{K}{2} = \frac{K}{2}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$y[n] = \text{---}$$

Causal LTI

$x[n] = 0$, for $n \leq -1$

$\therefore y[n] = 0$, for $n \leq -1$

$\Rightarrow y[-1] = 0$

← Check yourselves!